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Higher Energy Solutions in the Theory of Phase Transitions: A Variational Approach

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We establish the existence of a higher-energy solution to the vector Ginzburg–Landau equation with a triple-well potential on a bounded and smooth domain on the plane. This solution is obtained by a linking argument. In implementing this variational approach we make several considerations on the dynamics of the negative gradient flow. In particular, we use the Conley index to contruct a suitable one-dimensional invariant set. This solution has Morse index two in the nondegenerate case. We discuss its structure in connection with the so-called triple-junction configurations.

1. INTRODUCTION

In this paper we study the system

$$\begin{aligned} -\varepsilon^2 \Delta u + \nabla W(u) &= 0 \qquad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

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where $u\colon \Omega \to \mathbb{R}^2$, $\Omega \subset \mathbb{R}^2$ is a bounded regular domain, $W\colon \mathbb{R}^2 \to \mathbb{R}^+$ is a triple well potential in $\mathbb{R}^2$, v is the outer normal to the boundary and ε is a small parameter. Equation (1) arises naturally as the Euler–Lagrange equation of the following functional:

$$E_\varepsilon(u) = \int_\Omega \left(\frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{\varepsilon} W(u) \right) dx. \tag{2}$$

This problem appears in several physical contexts, including the study of motion of grain boundaries in alloys, capillary phenomena and certain models for phase transitions in the so-called mean-field approximation (see [1, 15, 16, 18, 27, 28] and references therein). Its solutions correspond to stationary points of the associated gradient flow [3, 11].

One of the most important features of problem (1) is that it allows for triple junction configurations, i.e., solutions for which the boundaries separating two phases meet at one point. In fact, this is the reason why it was introduced (see [28] for a discussion in the context of capillarity). Indeed, by considering a scalar potential with three minima a, b and c (see Fig. 1), it

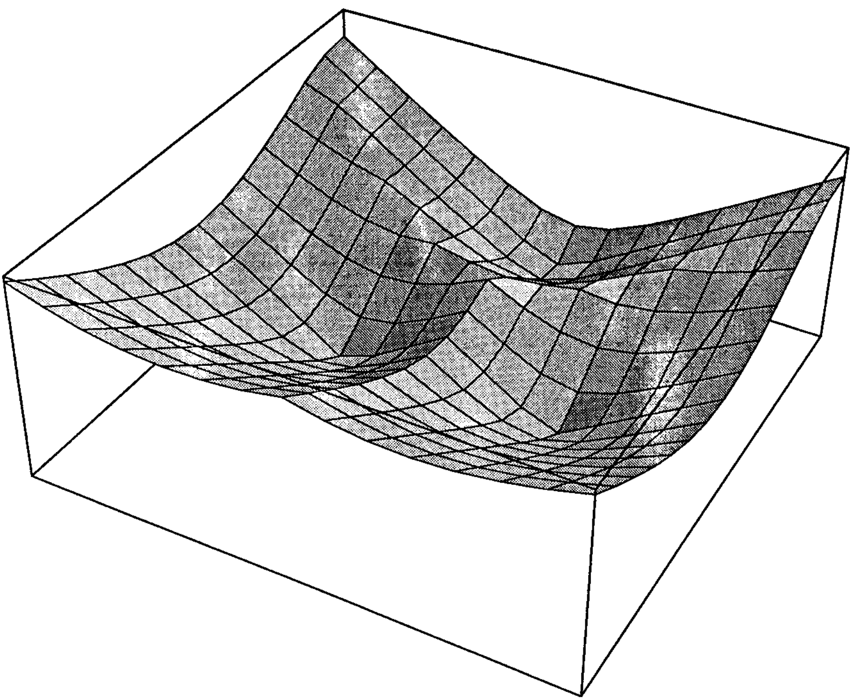

FIGURE 1

is clear that $u \equiv a$, $u \equiv b$, and $u \equiv c$ are spatially homogeneous solutions of the problem and that only transtition layers (*wetting transtitions*) between ab and bc can occur.

The scalar potential problem and the two-well vector valued potential problem have been considered by many authors. We refer the reader to [17] for a more complete list of references. The vector valued case, in general, and the vector valued case with three wells have also been considered [3, 4, 21, 23].

In [3] Bronsard and Reitich proved a local in time existence result for a system of three interfaces moving by mean curvature, which is the singular limit of (1) when $\varepsilon \to 0$.

In [2] Baldo establishes the Γ limit of (1) generalizing results in [12, 14, 21, 22]. Using also Γ-convergence methods, Sternberg and Ziemer were able to prove existence of local minimizers in L^1 for clover-shaped domains for ε sufficiently small [23].

The limiting case, in the L^1 norm, corresponds to the geometric partition problem of a domain in three regions, minimizing the length of the boundary separating them. Recently, in [4], Bronsard, Gui and Schatzman also showed the existence of a solution of Eq. (1) with a symmetric potential in $\mathbb{R}^2$ by using variational techniques and suitable estimates for certain solutions on bounded domains, which are used as approximations to the solution on the plane. In the last two works, [4] and [23], the fact that the solution corresponds to a triple-junction configuration is made in a very precise way. For instance, in [23] an additional necessary condition for a local minimizer u with a triple junction of (2) arises naturally,

$$\frac{\sin(\gamma_1)}{d(b,c)} = \frac{\sin(\gamma_2)}{d(a,c)} = \frac{\sin(\gamma_3)}{d(a,b)},$$

where d denotes the metric defined in Remark 1 below, γ_1 denotes the angle formed by the boundaries of $u^{-1}(a)$ and similarly for γ_2 and γ_3. These formulae are well known in the theory of phase transitions (e.g., in grain boundary motion [15] or simple fluid phases in equilibrium [5]).

The question we want to address here is whether such triple-junction configurations exist for a general bounded domain. In order to do that, we formulate the problem variationally and obtain a higher energy solution for regular bounded domains under appropriate assumptions. However, we are only able to establish that this higher energy solution has a triple-junction structure in a very qualitative way. We will make precise our assumptions on the potential (see conditions (i)–(v) in Section 2), but roughly speaking, the main result can be stated by saying that for an admissible triple-well potential W there exists a solution of (1) with Morse index 2 (the precise statement is given in Theorem 3.1).

Before proceeding with the proof, we describe the basic ideas involved, not only for the sake of clarity, but also because we believe that they might be applied in different contexts. In fact, what we do is to construct an invariant curve (under the negative gradiant flow of E_ε) containing a critical point for which a local linking structure can be established. The rest of the proof is very similar to the ones in [24] or [26], where higher energy solutions are found for semilinear scalar equations.

So consider a potential W whose minima are $a, b, c \in \mathbb{R}^2$. These points correspond to minima of the energy E_ε, i.e., constant solutions of Eq. (1). By the mountain pass lemma we can obtain three more critical points, one for each pair of equilibra: u_{ab}, u_{ac}, u_{bc}. For the class of potentials we consider, these points are different. Then we show that there exist two connecting orbits of the negative gradient flow joinning u_{ab} to a and b respectively. A similar argument provides us with connections between u_{ac} and a, c and u_{bc} and b, c.

The result is an invariant "triangle" in function space (see Fig. 2).

For proving the existence of the connecting orbits we use Conley index techniques. Specifically, we apply a result due to Conley and Smoller.

In the nondegenerate case, the Morse index of the mountain pass is one. This also gives the dimension of the unstable manifold. Let u_{ab} be the

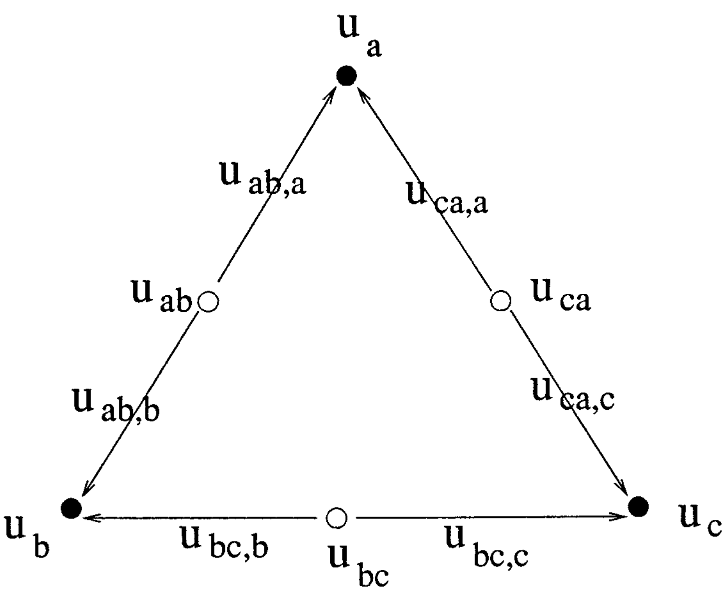

FIGURE 2

mountain pass with maximum energy, that is $E_\varepsilon(u_{ab}) \geqslant E(u_{ac}) \geqslant E(u_{bc})$. Since the unstable manifold is one dimensional and by construction $E_\varepsilon(u))$ decreases along the connecting orbits joinning u_{ab} to u_a and u_b, these curves determine the unstable direction. This means that in any other direction E_ε should increase, providing a local linking structure. By considering the set S of all surfaces $s(t)$ spanned by the invariant triangle, we can define

$$c_\varepsilon = \inf_{s \in S} \max_t E_\varepsilon(s(t)),$$

where $t = (t_1, t_2)$ gives the parametrization of S. By the generalized minimax principle, c_ε is a critical value and the local linking structure guarantees that $c_\varepsilon > E(u_{ab})$ and therefore that there exists a critical point u_ε, different from the previous ones.

Having established the existencde of the higher-energy solution, we study its properties. In particular, we would like to establish the behavior when $\varepsilon \to 0$ and to verify that it corresponds to a triple-junction configuration. We are not able to do this in a precise way, but in the nondegenerate case, we can apply techniques due to Fang and Ghoussoub [10] to characterise the Morse index of u_ε. We conjecture that when $\varepsilon \to 0$ this information and the fact that it is obtained via a specific minimax principle should be enough to establish the triple-junction structure. This is based on the analogy with some scalar equations, for which a precise connection between the Morse index and the nodal properties of solutions can be rigorously shown [6]. The existence of such a solution can be explicitely established for specific potentials [28]. In what follows we outline the content of the paper. In Section 2 we formulate the problem variationally and construct the invariant triangle described in this introduction. In Section 3 we implement the minimax argument using the local linking structure, proving the main result. We also present some results for the scalar case where assumptions on the symmetry of the potential allows us to make a similar construction and discuss the relation with previous work by Struwe and Wang. In Section 4 we discuss several open questions, in particular related to the structure of the interfaces of the solution and provide some concluding remarks.

2. VARIATIONAL AND DYNAMICAL CONSIDERATIONS

As pointed out in the introduction, we are interested in finding a higher energy critical point of (2). We begin by describing in detail the kind of potential W we will consider. We would like to emphasize, as will be evident from the proofs, that our assumptions are not optimal and in several cases can be relaxed. We prefer to present the result and ideas with a minimum of technical complications. So let $W: \mathbb{R}^2 \to \mathbb{R}$ be a smooth function. We will assume that the following conditions are satisfied:

(i) $W(u_1, u_2) \geqslant 0$ for all $u = (u_1, u_2) \in \mathbb{R}^2$.

(ii) Let $P_1 = a$, $P_2 = b$ and $P_3 = c \in \mathbb{R}^2$ be three different points such that $W(a) = W(b) = W(c) = 0$, i.e., global minima of W. We will also suppose that they are nondegenerate. For simplicity we will take these as the only minima of the potential.

(iii) In addition, we will assume that it is coercive, uniformly in u:

$$W(u) \geqslant C\,|u|^2,$$

if $|u| \geqslant R_0$, and that minus the gradient flow of W points to the interior of the ball $B_{R_0}(0)$ provided R_0 is sufficiently large.

Conditions (i)–(iii) define a triple-well structure for W. For technical reasons, we have to consider curves joining pairs of equilibria as well as the flow $\varphi_t(u)$ in $\mathbb{R}^2$ generated by $-\nabla W$. So we will assume that there exist curves $\gamma_{ij}: [0, 1] \to \mathbb{R}^2$, $i < j$, with $\gamma_{ij}(0) = P_i$, $\gamma_{ij}(1) = P_j$ such that

(iv) $\gamma_{ij} \subset C_{ij}$, where C_{ij} is a convex open set in $\mathbb{R}^2$, and such that the flow always points inwards along its boundary.

(v)
$$C_{12} \cap C_{13} \subset B_r(P_1),$$
$$C_{12} \cap C_{23} \subset B_r(P_2)$$

and

$$C_{13} \cap C_{23} \subset B_r(P_3),$$

where $r > 0$ is sufficiently small (see Fig. 3).

Remark 1. Although this is a technical restriction, the connections between minima of the potential and their length as curves on the graph of W have an important physical meaning; namely they are related to the energy of the transitions, surface tension, etc. A related quantity, the degenerate Riemannian metric

$$d(\xi_1, \xi_2) = \inf \left\{ \int_0^1 W^{1/2}(\gamma(t))\,|\gamma'(t)|\,dt : \right.$$
$$\left. \gamma \in C^1([0, 1], \mathbb{R}^2),\ \gamma(0) = \xi_1,\ \gamma(1) = \xi_2 \right\},$$

plays a fundamental role in the limit problem when $\varepsilon \to 0$ (see [23, Theorem 1.1]).

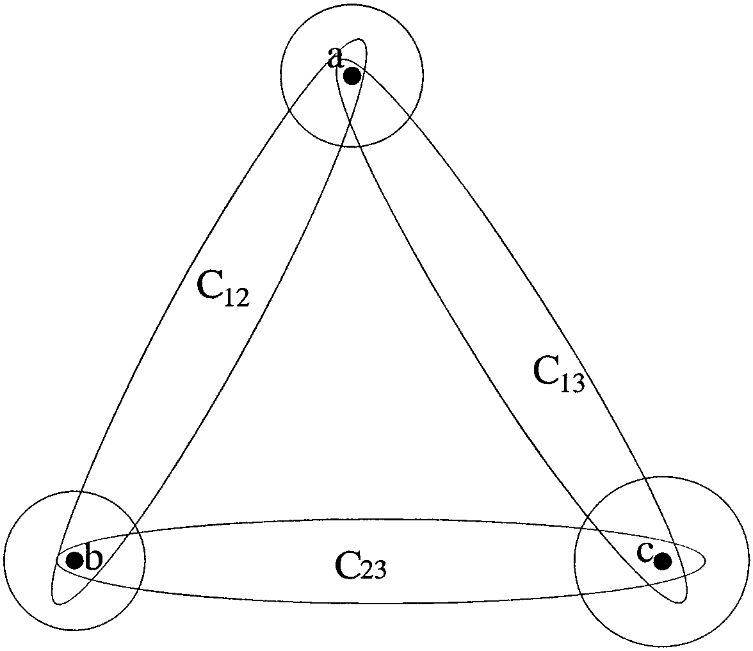

FIGURE 3

As an example, a potential for which the connections between equilibria (given by the steepest descent curves) are straight lines will satisfy (iv) and (v).

Moreover, in (v) any δ-neighbourhood of this straight line will do, provided that the steepest descent path is isolated and δ is sufficiently small. For instance, we can consider a regularization of the potential

$$W(u) = \operatorname{dist}(u, \{P_1, P_2, P_3\})^2.$$

The points P_i define a triangle on the plane. We take $P_1 = (-1, 0)$, $P_2 = (1, 0)$ and $P_3 = (0, b)$. The perpendicular bisectors of the sides of the triangle intersect at the point $\bar{P} = (0, (b^2 - 1)/2b)$, which is in the interior of the triangle provided $b^2 > 3$. These bisectors divide the plane into three regions and one expects $\bar{P}$ to be the point determining the triple junction structure. This is basically what is proved in the problem studied in [28]. Under condition (iii), it is standard that (2) satisfies the Palais–Smale (PS) condition [25].

In order to make sure we can obtain three different mountain passes, we need a result on invariant regions, as presented in [20].[4]

First we recall some definitions and concepts from the theory of dynamical systems.

We denote by $\Phi(t, u_0)$ the flow at time t with initial condition u_0 generated by minus the gradient flow of E_ε, that is the solution of

$$u_t = \varepsilon^2 \Delta u - \nabla W(u), \tag{3}$$

with

$$u(0, x) = u_0(x). \tag{4}$$

We will define, given a subset D in $\mathbb{R}^2$, a subset of functions in $L^2(\Omega)$, which we denote by the same letter, as

$$D = \{u \in L^2(\Omega) \mid u(x, y) \in D \text{ a.e in } \Omega\}.$$

DEFINITION 1. A closed subset $\Sigma \subset \mathbb{R}^2$ is called (positively) invariant for the flow generated by (3) and (4), if any solution $v(x, y, t)$ having all of its boundary and initial values in Σ, satisfies $v(x, y, t) \in \Sigma$ a.e. in Ω and for all $t \in [0, \delta)$, where it is assumed that solutions exist at least up to $t < \delta$.

For the variational problem, if we consider E_ε defined in $W_2^2(\Omega)$, it is well known that the negative gradient generates a global semiflow.

We also have the following a priori estimates for $t, s \geqslant 0$,

$$\|u(t)\|_{W_2^2(\Omega)} \leqslant C(1 + \|u_0\|_{W_2^2(\Omega)}),$$

and

$$\|u(t) - u(s)\|_{L^2(\Omega)} \leqslant C(1 + \|u_0\|_{W_2^2(\Omega)})\, |t - s|^\nu,$$

where $0 \leqslant \nu \leqslant 1$, with C and ν independent of t, s and u_0 ([20], pp. 504 ss). Finally, we quote several well known results that allow us to establish the existence of a local flow, so that we can apply Conley index theory (for the details we refer to Chapter 23, paragraph D, point 5 in [20]).

Let Γ be the Banach space of continuous curves

$$\gamma\colon \mathbb{R} \to L_2(\Omega),$$

[4] In fact, the result quoted below is presented in [20] for only one space dimension. However, as it is pointed out in this work, the result is also valid for several space dimensions on bounded domains and general boundary conditions (see [7] for the proof in the general case).

endowed with the compact-open topology. Then we define a continuous flow, the translation flow on Γ, setting for every $\sigma \in \mathbb{R}$

$$(\gamma \cdot \sigma)(t) = \gamma(\sigma + t),$$

for all $t \in \mathbb{R}$, $\gamma \in \Gamma$. We assume that D is positively invariant, that is, if $u(t)$ is a solution of (3) and (4) with $u_0 \in D \cap W_2^2(\Omega)$, then it remains in this set for all $t > 0$.

We have the following lemma (Lemma 23.33 in [20])

LEMMA 1. *Let Γ be the space of continuous curves γ: $\mathbb{R} \to L_2(\Omega)$ with the compact-open topology and*

$$\begin{aligned} X = \{\gamma \in \Gamma : \gamma(t) \in W_2^2(\Omega),\ \gamma(0) \in \Sigma \cap W_2^2(\Omega),\ \|\gamma(t) - \gamma(s)\|_{L^2} \\ \leqslant C'_\gamma\, |t - s|^\nu \text{ and } \|\gamma(t)\|_{W_2^2(\Omega)} \leqslant C_\gamma\ \forall s, t \in \mathbb{R}, t \geqslant 0, \\ \gamma \text{ solves } (3) \text{ with initial condition } \gamma(0)\}, \end{aligned}$$

where C_γ, C'_γ depend on γ but not on t.

The embedding $X \subset \Gamma$ is compact, thus X is a locally compact subset of Γ. This implies that X is a local flow in Γ.

We then have the following result, which is a particular case of Corollary 14.8 in [20]:

LEMMA 2. *Any convex region Σ in which $-\nabla W$ points into Σ on $\partial\Sigma$ is invariant for* (3).

Using this we can prove:

PROPOSITION 1. *There exist at least three distint critical points of mountain pass type u_{ab}, u_{ac}, u_{bc} for the functional E_ε.*

Proof. Notice that by Lemma 2 and (iv), C_{ij}, $i \neq j$, are invariant regions for Φ_t in X. Therefore we can apply the mountain pass theorem in each one. Indeed $u_i \equiv P_i$, $i = 1, 2, 3$, are local minima and $E_\varepsilon(u_i) = 0$. Moreover by (i) they are nondegenerate. Then

$$c_\varepsilon^{ij} = \inf_{\gamma \varepsilon \Gamma_{ij}} \max_{t \varepsilon [0, 1]} E_\varepsilon(\gamma(t))$$

is a critical value of E_ε, where

$$\Gamma_{ij} = \{\gamma \in C([0, 1], X \cap C_{ij}) \mid \gamma(0) = u_i, \gamma(1) = u_j\}.$$

Notice that the path

$$\gamma_{ij}(t) = tu_j + (1-t)\, u_i$$

is in Γ_{ij}. However, if we evaluate the energy along this path, it becomes unbounded as ε goes to zero. Since we are interested in the behavior of solutions for ε arbitrarily small, it is necessary to prove that the energy of the mountain pass solutions that we obtain remains bounded. In fact, this also rules out the possibility that these solutions are constants corresponding to the "mountain passes" of the potential W itself, since for these spatially homogeneous solutions the energy tends to infinity as $\varepsilon \to 0$.

Choose the rectangle $Q = [\alpha, \beta] \times [\gamma, \delta]$ in such a way that $\Omega \subset Q$ and $\operatorname{dist}(\partial Q, \partial\Omega) \geqslant 1$. Take a function $\phi(x) \in C_0^\infty(\mathbb{R})$ such that

1. $0 \leqslant \phi(x) \leqslant 1$,
2. $\phi(x) = 0$ for $x \in (-\infty, 0) \cup (1, \infty)$.

Let $\xi(t) = \alpha + (\beta - \alpha)\, t$, for $0 \leqslant t \leqslant 1$ and consider the path in function space defined by

$$\gamma(t)(x, y) = \phi\left(\frac{x - \xi(t)}{\varepsilon}\right) a + \left[1 - \phi\left(\frac{x - \xi(t)}{\varepsilon}\right)\right] b.$$

Then $\gamma(t) \in C^\infty(Q)$ for each $t \in [0, 1]$. Notice also that $\gamma(t)(x, y) = a$ for $x \leqslant \xi(t) - \varepsilon$ and $\gamma(t)(x, y) = b$ for $x \geqslant \xi(t) + \varepsilon$. If we take $a = (a_1, a_2)$ and $b = (b_1, b_2)$, we obtain that

$$\nabla(\gamma(t))(x, y) = \frac{1}{\varepsilon}\begin{pmatrix} \phi'\left(\frac{x - \xi(t)}{\varepsilon}\right)(a_1 - b_1) & 0 \\ \phi'\left(\frac{x - \xi(t)}{\varepsilon}\right)(a_2 - b_2) & 0 \end{pmatrix}.$$

It follows that $\nabla(\gamma(t)) = 0$ outside the strip $|x_1 - \xi(t)| \leqslant \varepsilon$. Moreover, $W(\gamma(t))(x_1, x_2)) = 0$ outside this strip too. We conclude that

$$E_\varepsilon(\gamma(t)) = \int_\Omega \frac{\varepsilon}{2}\, |\nabla(\gamma(t))|^2 + \frac{1}{\varepsilon}\, W(\gamma(t))\, dx_1\, dx_2 = O(1),$$

as $\varepsilon \to 0$.

Now we show that the critical points u_{ij} corresponding to c_ε^{ij} are distinct if r is sufficiently small.

Indeed, observe that if $u_{ij} \in B_r(P_i)$, by multiplying equation (1) by u_{ij} and integrating by parts we have

$$\varepsilon^2 \int_\Omega |\nabla u_{ij}|^2 = \int_\Omega \nabla W(u_{ij})\, u_{ij}$$

and since $u_{ij} \in B_r(P_i)$, using the fact that $W \in C^1$ and $\nabla W(P_i) = 0$, we have

$$\int_\Omega |\nabla u_{ij}|^2 = o(r) \qquad \text{for fixed } \varepsilon.$$

But this implies

$$E(u_{ij}) = o(r).$$

However, by the mountain pass theorem we know that there is a constant ρ_i, independent of r such that $E_\varepsilon(u_{ij}) \geqslant \rho_i$.

Therefore $u_{ij} \notin C_{ij} \cap C_{ik}$. We conclude that $u_{ij} \neq u_{jk}$ for the three different choices of the subindices and r sufficiently small, and identify u_{ij} with u_{ab}, u_{ac}, u_{bc}, in the obvious way.

We now prove the existence of the connecting orbits as described in the introduction.

PROPOSITION 2. *Given W as before, that is, satisfying conditions* (i) *to* (v), *and assuming*:

1. u_{ab}, u_{ac}, u_{cb} *are nondegenerate*

2. *the mountain pass solution u_{ij} is the nontrivial, i.e. nonconstant, critical point with least energy contained in C_{ij}, then, there exist full orbits $u_{ij,\,i}\, u_{ij,\,j}$ (that is solutions of Eq.* (3)) *defined for all $t \in \mathbb{R}$ such that*

$$\begin{cases} u_{ij,\,i}(t, x) \to u_i \\ u_{ij,\,j}(t, x) \to u_j \end{cases}$$

when $t \to \infty$ and

$$\begin{cases} u_{ij,\,i} \to u_{ij} \\ u_{ij,\,j} \to u_{ij}, \end{cases}$$

when $t \to -\infty$.

Remark 2. Assumptions (1) and (2) are taken here for convenience and in order to apply Conley index theory as simply as possible. However, we observe that the existence of a connecting orbit from a critical point of mountain pass type to the minimum "in the valley" should be a generic property (cf. [19], where the existence of such orbit is proved using also the Conley index and [20, Theorem 24.1] which provides a similar result under slightly different assumptions). We refer to the comments at the end of Section 4.

First we recall the definitions and results we need (for a detailed treatment we refer to [20, Chap. 22–24]. Although Rybakowski [19] has generalized Conley's ideas to the infinite dimensional case, we can here rely on the treatment given by Smoller. This is due to the fact that the local flow determined by (3) and (4) can be embedded in a flow under our assumptions on W as was already pointed out in this section.

DEFINITION 2. 1. The closure of a bounded open set $N \subset X$ is called an isolating neighborhood for Φ if for each $u \in \partial N$, there is a $t \in \mathbb{R}$ such that $\Phi(t, u) \notin N$.

2. A closed invariant set I is called an isolated invariant set if it is the maximal invariant set in some isolating neighborhood.

DEFINITION 3. Let $S \subset X$. Given $\delta > 0$, we define h_δ: $S \times (-\delta, \delta) \to X$ by $h_\delta(u, t) = \Phi(t, u)$. If for some δ, h_δ is a homeomorphism with open range in X, then S is called a local section.

DEFINITION 4. Let B be the closure of an open set in X and S^+, S^- disjoint local sections satisfying

1. $[cl(S^\pm) \backslash S^\pm] \cap B = \phi$,
2. $\Phi((-\delta, \delta), S^-) \cap B = \Phi([0, \delta), (S^- \cap B))$.
3. $\Phi((-\delta, \delta), S^+) \cap B = \Phi((-\delta, 0], (S^+ \cap B))$
4. if $u \varepsilon\, \partial B \backslash (S^+ \cup S^-)$, then there exist $\varepsilon_1 < 0$ and $\varepsilon_2 > 0$ such that $\Phi([\varepsilon_1, \varepsilon_2], u) \subset \partial B$ and $\Phi(\varepsilon_1, u) \in S^-$, $\Phi(u, \varepsilon_2) \in S^+$,

then B is called an isolating block.

For an isolating block we define the following sets

$$b = \partial B$$

$$b^+ = B \cap S^+ \subset b$$

$$b^- = B \cap S^- \subset b.$$

DEFINITION 5. With this notation, we define the Conley index or the homotopy index of an isolated invariant set I as $h(I) = [B/b^+]$, that is, the homotopy type of the pointed space B/b^+, where B is an isolating block for I.

Remark 3. It is proved in [20] that the homotopy index is well defined. In particular, it does not depend on the isolating block.

We also have the following fact:

LEMMA 3. *Suppose that u_0 is an isolated rest point in a local flow X of a gradient like equation. Then u_0 is an isolated invariant set. In particular, if u_0 is a nondegenerate critical point with Morse index k, then $h(u_0) = S^k$.*

The result that allows us to conclude the existence of an orbit connecting two rest points is the following.

THEOREM 1. *Let $\dot{u} = f(u)$ be a gradient system in an isolating block N, containing precisely two rest points u_1, u_2 of f, not both of which are degenerate. Let $S(N)$ be the maximal invariant set in N. If $h(S(N)) = \bar{0}$ (the homotopy type of a point), then there is an orbit of f connecting the two rest points.*

We can now prove Proposition 2. By construction C_{ij} is an isolating neighbourhood in X and the negative gradient flow points inwards at each point of the boundary. Moreover, since we are assuming that u_{ij} is the least energy, nontrivial solution, we can further restrict the flow to $C_{ij} \cap \{u \mid E_\varepsilon(u) \leqslant c_\varepsilon^{ij}\}$. Let $V_i \subset C_{ij}$ be an isolating neighbourhood of u_i (one can take a ball of sufficiently small radius). Since u_i is a nondegenerate minimum we can construct an isolating block $B = \overline{C_{ij}/V_i}$, for which the flow is as seen in Fig. 4.

Then $b^+ = \partial V$ and (B, b^+) is an index pair. Moreover, B/b^+ has the homotopy type of a point, since V is contractible to a point. Thus the Conley index of S the maximal invariant set in B is $\bar{0}$. By Theorem 1 we conclude that there is an orbit connecting u_{ij} to u_j. In an analogous way, we conclude the existence of the other connecting trajectories.

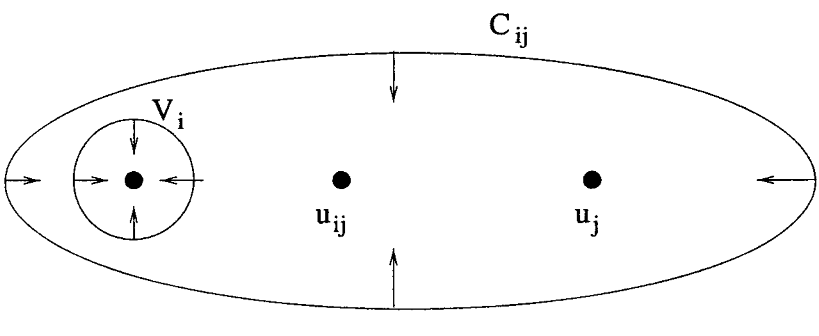

FIGURE 4

3. THE MINIMAX PROCEDURE

Now we can implement the minimax procedure in a straightforward way. The equilibria u_i, the mountain pass solutions u_{ij}, and the connecting orbits obtained in the previous section determine an invariant triangle T in function space. Consider the family of surfaces in X spanned by T:

$$S = \{ s \in C(\sigma, H^1(\Omega)) : s|_{\partial\sigma} = T \}$$

where σ is the triangle depicted in Fig. 5.

We define

$$c_\varepsilon = \inf_{s \in S} \max_{t \in \sigma} E_\varepsilon(s(t)).$$

THEOREM 2. *Assume that W satisfies* (i)–(iv) *and the hypotheses of Proposition 2.2, then there exists a solution of* (1) *with energy strictly greater than* $\max(E(u_{ij}))$. *Moreover, if this solution corresponds to a nondegenerate critical point of* E_ε, *its Morse index is two.*

Proof. The proof is divided into two steps: The fact that c^ε is a critical value follows from the general minimax theorem [25].

PROPOSITION 3. *Assume that $f \in C^1(E, \mathbb{R})$, where E is a Hilbert space and $\mathscr{F}$ is a family of functions from σ into E, which meet the following conditions*:

(i) *f satisfies the Palais-Smale condition.*

(ii) *$\dot{x} = -\nabla f(x)$ defines a global flow $\phi_t(x)$.*

(iii) *The family $\mathscr{F}$ is positively invariant under the flow, i.e., if $F \in \mathscr{F}$ then $\phi_t(F) \in \mathscr{F}$ for every $t \geqslant 0$.*

(iv) $-\infty < c(f, \mathscr{F}) < \infty$, *where*

$$c(f, \mathscr{F}) = \inf_{F \in \mathscr{F}} \sup_{s \in \sigma} f(F(s)).$$

Then $c(f, \mathscr{F})$ is a critical value of f.

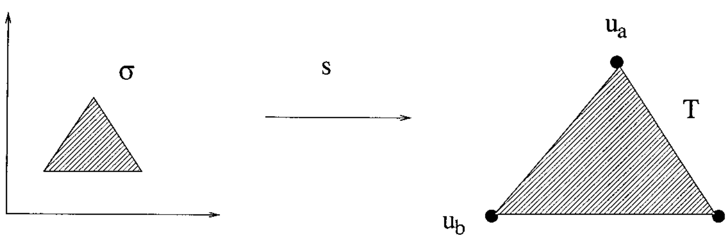

FIGURE 5

We now apply this result to our problem.

Conditions (i) and (ii) were established in the previous section.

Condition (iii) follows immediately from the fact that T consists of trajectories of the flow ϕ, either critical points or connecting trajectories, so that the set S is positively invariant under the flow. To see that this family is non-empty, just map $\partial\sigma$ into T in a continuous way and use Tietze's extension theorem to get a continuous map defined on σ with the desired properties. The left inequality in (iv) is obvious, since the functional we consider is positive. The right inequality is a consequence of the compactness of σ and the continuity of f and F. Moreover, that $c_\varepsilon > E(u_{ij})$ is a consequence of the local analysis in [26] and we omit it here fo the sake of brevity.

In order to prove the fact that the Morse index is two in the non-degenerate case, we apply a result due to Fang [9], which characterizes the Morse index of critical points obtained via a minimax principle (Theorem 4.6):

Proposition 4. *Let f be a C^2 functional on a Hilbert space E and consider a homotopic family $\mathscr{F}$ of dimension n with nonempty and closed boundary B. Let $\mathscr{F}^*$ be a family dual to $\mathscr{F}$. Suppose that*

$$c = \sup_{F^* \in \mathscr{F}} \inf_{x \in F^*} f(x) = \inf_{F \in \mathscr{F}} \sup_{x \in F} f(x)$$

is finite. Assume that f satisfies the PS condition and suppose that the critical points of f are nondegenerate. Then, there exists a critical point x with $f(x) = c$ and Morse index $m(x)$ satisfying

$$\min\{2, n\} \leqslant m(x) \leqslant n.$$

For the details we refer to the work by Fang. In our case, it suffices to point out that we can take $\mathscr{F}$ to be the class S. If $\sup_{x \in \partial B} f(x) < c$ it is shown in Chap. 3 of the same work that the set $\{f \geqslant c\}$ is dual to $\mathscr{F}$. Moreover, n in our case is equal to two. So we conclude the existence of a critical point with Morse index two. This concludes the proof of the main result. However, several observations are in order. We have assumed throughout that the mountain passes are nondegenerate. The nondegeneracy was used only in three situations: in Proposition 2.2 in order to prove the existence of the connecting trajectories; in the implementation of the minimax procedure, to show the local linking structure (see [26]) and in the conclusion about the Morse index of the higher energy solution. We point out that Dancer has studied the Conley index of degenerate critical points [8] and that the results establishing the existence of connecting orbits can be applied in this case too. Also, the local linking structure is established in [9] for the degenerate situation, although it involves a more technical application of the Morse lemma. Finally, Fang's results are also applicable in the degenerate case, obtaining inequalities for the Morse indices of critical points. Therefore, extensions of our result to the general case can be proved, although the statements and proofs would certainly be more technical.

We end this section with several remarks related to the scalar case, that is, when we have a single equation

$$\begin{aligned} -\varepsilon^2 \Delta u + \nabla W(u) &= 0 \qquad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{5}$$

where $u(x, y)$ takes real values and $W: \mathbb{R} \to \mathbb{R}$ and has, for example, a double well structure. To fix ideas, consider $W(u) = (1 - u^2)^2$. In this case we have two spatially homogeneous minima $u \equiv \pm 1$. We also have one mountain pass solution u_1. However, by the symmetry of the potential, we also have a second solution $-u_1$. Since they have the same energy, they cannot be connected by any trajectory of minus the gradient flow. By essentially repeating the same argument as in the proof of Proposition 2.2 we conclude the existence of connecting trajectories joinning each of these mountain pass type solutions to both minima. Thus we obtain an invariant loop in function space and can apply the same minimax argument. Notice that this is in fact a simple case of the method for finding a third solution of Wang [26], since our functional is bounded from below. By assuming further symmetries of Ω, one can implement other linking situations. The question is again what kind of interface structure develops when ε tends to zero.

4. FINAL REMARKS AND OPEN QUESTIONS

The question of the structure of the interfaces of the solution when ε tends to zero is a very important an interesting one. For the scalar case, much work has been done and, roughly speaking, it has been established that the limit case corresponds to the geometric problem of minimizing perimeter. In [17] it is proved that the limit of the interfaces, in the varifold sense, corresponds to a stationary varifolds. It would be interesting if this approach could be generalized to the vector valued case.

Another problem that seems relevant to us is the study of the multiplicity of solutions and the location of the interfaces, depending on the geometry of the domain, as well as the existence of solutions with several interfaces, that is, the equivalent of "multi-peaks" solutions in this context.

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